

Four-loop contributions to long-distance quantities in the two-dimensional nonlinear σ -model on a square lattice: revised numerical estimates*

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Abstract

We give the correct analytic expression of a finite integral appearing in the four-loop computation of the renormalization-group functions for the two-dimensional nonlinear σ -model on the square lattice with standard action, explaining the origin of a numerical discrepancy. We revise the numerical expressions of Caracciolo and Pelissetto for the perturbative corrections of the susceptibility and of the correlation length. For the values used in Monte Carlo simulations, $N = 3, 4, 8$, the second perturbative correction coefficient of the correlation length varies by 3%, 4%, 3% respectively. Other quantities vary similarly.

11.10.Gh; 11.10.Kk; 11.15.Ha; 12.38.Bx; 75.10.Hk

The two-dimensional nonlinear σ -model has been extensively studied since it is supposed to share with four-dimensional QCD the property of being asymptotically free. A lot of work has been devoted to the checking of renormalization-group predictions. In order to make accurate comparisons it is very important to have high-order perturbative predictions. In [1] the renormalization-group β - and γ -functions were computed for the standard action with nearest-neighbour interactions on a square lattice. The result was expressed in terms of a small number of basic lattice integrals that were computed numerically. In Ref. [2], using the coordinate space method [3], the lattice integrals that appear in the results of [1] were independently evaluated with higher numerical precision, and two of them were found grossly incorrect. As we shall discuss below, the large discrepancy found for one of them was due to the fact that its definition was not correctly given in Ref. [1].

Two of the present authors (B. A. and M. P.) have performed a complete independent check of the analytic part of the lattice calculation, by computing the two-point function in the presence of an external constant magnetic field to three loops. Their result is in agreement with that presented in Ref. [1]. However their computation allowed the discovery of an incorrect definition in Ref. [1]: the constant W_2 which was used in the computation of Ref. [1] does not have the definition given in App. A.1. Let us indicate with \widehat{W}_2 the definition appearing in App. A.1 of Ref. [1]:

$$\begin{aligned} \widehat{W}_2 \equiv \lim_{h \rightarrow 0} & \left[\int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 r}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 s}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 t}{(2\pi)^2} (2\pi)^2 \delta^2(q+r+s+t) \right. \\ & \times \frac{\sum_{\mu\nu} \sin q_\mu \sin r_\nu \sin s_\mu \sin t_\nu}{(\widehat{q}^2 + h)(\widehat{r}^2 + h)(\widehat{s}^2 + h)(\widehat{t}^2 + h)(\widehat{q+r}^2 + h)} \\ & \left. - \frac{1}{6} I(h)^3 + \frac{1}{8} \left(1 - \frac{1}{\pi} \right) I(h)^2 - \left(\frac{1}{32} + \frac{1}{16\pi^2} - \frac{1}{16\pi} - \frac{1}{2} R \right) I(h) \right], \end{aligned} \quad (0.1)$$

where $\widehat{p}^2 = \sum_\mu 4 \sin^2 p_\mu/2$, $I(h)$ is the integral of the propagator

$$I(h) \equiv \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \frac{1}{\widehat{p}^2 + h} = -\frac{1}{4\pi} \log \left(\frac{h}{32} \right) + O(h), \quad (0.2)$$

and $R \approx 0.0148430$ is a numerical constant introduced in [4]. In the calculation of Ref. [1], W_2 was instead defined as

$$W_2 = \widehat{W}_2 + \frac{85}{2304\pi^3} \zeta(3). \quad (0.3)$$

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This difference explains most of the discrepancy found in Ref. [2] for the numerical estimate of W_2 . Using [2]

$$\widehat{W}_2 = 0.0006923019(1), \quad (0.4)$$

we find

$$W_2 = 0.002122552, \quad (0.5)$$

which should be compared with $W_2 = 0.00221$ reported in Ref. [1]. The remaining small discrepancy was due to numerical problems that were not understood at the time. We have completely revised the programs and now we find $W_2 = 0.002122(1)$ in agreement with (0.5). It follows that, while the analytic expressions of Ref. [1] are correct, the numerical results for the perturbative corrections presented in Refs. [1,2] must be revised. Below we give the numerical expressions for the perturbative constants a_2 , b_2 , c_3 , and $b_2^{(n)}$ (see Ref. [1] for definitions), using the numerical estimates of Ref. [2] and the estimate (0.5):

$$a_2 = \frac{1}{(N-2)^2} (0.0444 + 0.0216N + 0.0045N^2 - 0.0129N^3), \quad (0.6)$$

$$b_2 = \frac{1}{(N-2)^2} (0.1316 + 0.0187N - 0.0202N^2 - 0.0108N^3), \quad (0.7)$$

$$c_3 = \frac{N-1}{(N-2)^3} (0.0121 + 0.0122N - 0.0070N^2 - 0.0092N^3 + 0.0041N^4), \quad (0.8)$$

$$b_2^{(n)} = \frac{1}{(N-2)^2} [0.0912 + 0.0446N + 0.0093N^2 - 0.0257N^3 \\ + n(n+N-2)(-0.0363 - 0.0187N + 0.0149N^2) + 0.0041n^2(n+N-2)^2]. \quad (0.9)$$

For $N = 3$ we have $a_2 = -0.1969$, $b_2 = -0.2853$, $c_3 = 0.1346$: with respect to the previous estimates of Ref. [1], a_2 , b_2 , and c_3 vary by 3%, 5%, and 1% respectively. The difference is very small and does not change the conclusions of the papers that compared Monte Carlo results with the perturbative predictions [5–9].

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